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LETTER TO THE EDITOR

Directed lattice animals and the Lee–Yang edge singularity

John L Cardy
Department of Physics, University of California, Santa Barbara, California 93106, USA

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Abstract. The exponents $\nu_\perp$, γ of directed animals in d dimensions are shown to equal those for the Lee–Yang problem in $d-1$ dimensions. For $d=2$ we obtain the exact results $\nu_\perp=\theta=\frac{1}{2}$, and for $d\leqslant 7$ the scaling relation $\theta=(d-1)\nu_\perp$.

The problem of directed lattice animals has attracted recent theoretical attention (Redner and Yang 1982, Redner and Coniglio 1982, Dhar *et al* 1982, Day and Lubensky 1982). In this letter we point out that this problem, when formulated in field-theoretic terms, is equivalent to the critical dynamics of an Ising model in an imaginary field, the so-called Lee–Yang problem (Fisher 1978). The theory of dynamic critical phenomena then implies that the exponents $\nu_\perp$ and γ of directed animals are the static exponents of the Lee–Yang problem in one less dimension. For $d=2$, this implies the exact results $\nu_\perp=\frac{1}{2}$, $\gamma=\frac{3}{2}$ and $\theta=2-\gamma=\frac{1}{2}$, in agreement with numerical results (Redner and Yang 1982, Dhar *et al* 1982). Since there is only one independent exponent in the Lee–Yang problem, we may also infer a scaling relation $\theta=(d-1)\nu_\perp$. In addition, Parisi and Sourlas (1981) have shown that the exponents for the Lee–Yang problem in $d-1$ dimensions are the same as those for undirected animals in $d+1$ dimensions. We therefore have an interesting correspondence between these three models.

We first rederive the field theory of directed animals (Day and Lubensky 1982) in a systematic way, using a method similar to that employed for directed percolation (Cardy and Sugar 1980). Consider a lattice with a preferred direction (labelled by $r_\parallel$) with respect to which bonds are oriented. Introduce commuting pseudospins $a(\boldsymbol{r})$, $\bar{a}(\boldsymbol{r})$ at each site $\mathbf{r}$, which satisfy the algebra $a^2=a$, $\bar{a}^2=0$, and an operation Tr satisfying $\mathrm{Tr}\,a=0$, $\mathrm{Tr}\,\bar{a}=\mathrm{Tr}\,\bar{a}a=1$. The expression

$$\Phi(x)=\mathrm{Tr}\,\bar{a}(0)\prod_{(\boldsymbol{r},\boldsymbol{r}')}\{1+x\bar{a}(\boldsymbol{r}')a(\boldsymbol{r})\} \tag{1}$$

where the product is over all directed nearest-neighbour pairs $(\boldsymbol{r}\to\boldsymbol{r}')$ gives the generating function $\Sigma_n A(n)x^n$, where $A(n)$ is the number of directed animals, with no closed loops, rooted at $\boldsymbol{r}=0$. To convert (1) into a field theory we exponentiate the expression in braces and write this as a Gaussian integral:

$$\exp\Big(x\sum_{\boldsymbol{r}}\bar{a}(r')V(\boldsymbol{r}'-\boldsymbol{r})a(\boldsymbol{r})\Big)=\int\mathscr{D}\psi\mathscr{D}\bar{\psi}\,\exp\Big(\sum_{\boldsymbol{r}}(x^{-1}\bar{\psi}V^{-1}\psi+\mathrm{i}\bar{a}\bar{\psi}+\mathrm{i}a\psi)\Big). \tag{2}$$

The operation Tr can now be performed. In the continuum limit we obtain a field theory whose action is

$$A[\bar{\psi}, \psi] = -\int \mathrm{d}^d r \, [x^{-1}\bar{\psi}V^{-1}\psi + \ln(1 + \mathrm{i}\bar{\psi}\mathrm{e}^{\mathrm{i}\psi})] \tag{3}$$

where the Fourier transform of V^{-1} is $\tilde{V}^{-1} = b_0 + \mathrm{i}b_1 q_\parallel + b_2 q_\perp^2 + \ldots$. If we now shift the fields according to $\psi = \phi + \text{constant}$, $\bar{\psi} = \bar{\phi}$, to remove the linear term in $\bar{\phi}$, and expand in ϕ, $\bar{\phi}$ the action has the form

$$A[\bar{\phi}, \phi] = \int \mathrm{d}^d r \, [\bar{\phi}(\partial_\parallel - \partial_\perp^2)\phi + c_1\bar{\phi}\phi - c_2\bar{\phi}^2 + \mathrm{i}c_3\bar{\phi}\phi^2 + \ldots] \tag{4}$$

where the c_i are complicated functions of x, and we have kept only the most relevant terms. This expression is of the form written down by Day and Lubensky (1982).

Consider now a continuous spin Ising model in an imaginary field with Hamiltonian

$$H[\psi] = \int \mathrm{d}^{d-1} r \, [\tfrac{1}{2}(\partial_\perp \psi)^2 + \tfrac{1}{2}r_0\psi^2 + \tfrac{1}{4}u\psi^4 + \mathrm{i}h\psi]. \tag{5}$$

The dynamics of this model, as given by the Langevin equation

$$\dot{\psi} = -\Gamma \partial H/\partial\psi + \eta \tag{6}$$

where η is a random noise, may be expressed in terms of a dynamic functional (Martin *et al* 1972)

$$\int \mathscr{D}\psi\mathscr{D}\bar{\psi} \exp\left(-\int \mathrm{d}^d r \, [\bar{\psi}(\partial_\parallel - \partial_\perp^2)\psi + \mathrm{i}h\bar{\psi} + r_0\bar{\psi}\psi + u\bar{\psi}\psi^3 - \Gamma\bar{\psi}^2]\right) \tag{7}$$

where we have introduced a response field $\bar{\psi}$ and identified 'time' with $r_\parallel$. If we now remove the linear term in $\bar{\psi}$ as above, and keep only relevant terms, we get an action of the form (4) with $c_2 = \Gamma$.

The response function $\langle\psi(\boldsymbol{r})\bar{\psi}(0)\rangle$ is just the function $G_{\bar{\psi}\psi}$ introduced by Day and Lubensky (1982). From the general theory of dynamics, $G_{\bar{\psi}\psi}(q_\perp, q_\parallel = 0)$ must equal the static correlation function of the Hamiltonian (5). Therefore the exponents γ and $\nu_\perp$, defined by $G_{\bar{\psi}\psi}(q_\perp = q = 0) \propto |x - x_\mathrm{c}|^{-\gamma}$ and $\xi_\perp \propto |x - x_\mathrm{c}|^{-\nu_\perp}$, are those of the Lee–Yang problem (5), in $d-1$ dimensions.

For the Ising model in $d = 1$, with exchange interaction J and external field $\mathrm{i}h$, the free energy per site f and the correlation length are respectively

$$f = -\ln \lambda_+, \qquad \xi = [\ln(\lambda_+/\lambda_-)]^{-1} \tag{8}$$

where

$$\lambda_\pm = \mathrm{e}^J \cos h \pm (\mathrm{e}^{-2J} - \mathrm{e}^{2J} \sin^2 h)^{1/2}. \tag{9}$$

From this we conclude that $\nu_\perp = \frac{1}{2}$, $\gamma = \frac{3}{2}$, giving the results quoted in the abstract. Since the free energy for the Lee–Yang problem scales as $(h - h_\mathrm{c})^{1+\sigma}$ with $1 + \sigma = \theta = 2 - \gamma$, by hyperscaling (Fisher 1978) the correlation length scales as $(h - h_\mathrm{c})^{-(1+\sigma)/(d-1)}$. This gives the second scaling relation. We conclude by displaying the full set of relations for both ordinary animals (Parisi and Sourlas 1981) and directed animals:

$$\text{Ordinary:} \qquad \theta(d) = 2 + \sigma(d-2) = 1 + (d-2)\nu(d), \tag{10}$$

$$\text{Directed:} \qquad \theta_\mathrm{D}(d) = 1 + \sigma(d-1) = (d-1)\nu_{\perp\mathrm{D}}(d), \tag{11}$$

from which we see that $\nu_{\perp D}(d) = \nu(d+1)$. The critical exponent $\nu_\parallel$ of directed animals requires knowledge of the dynamics of the Lee–Yang problem. This appears non-trivial, even in one dimension.

In table 1 we test the relations (11) against numerical work on the Lee–Yang problem and directed animals. Although the results on the latter have sizeable quoted errors, there appears to remain significant discrepancies in $d = 3$ and 4, which we do not at present understand. The relations are correct at $d = 2$ and to first order in the $(7-d)$ expansion (Day and Lubensky 1982, Fisher 1978).

Table 1. Test of scaling relations $\theta = 1 + \sigma(d-1) = (d-1)\nu_\perp$.

	$1+\sigma(d-1)$, Lee–Yang		Directed animals		
d	Series[a]	ε expansion[b]	θ[c]	$(d-1)\nu_\perp$[c]	$(d-1)\nu_\perp$[d]
2	0.5[e]	0.507 ± 0.01	0.53 ± 0.01	0.500 ± 0.003	0.56
3	0.837 ± 0.003	0.845 ± 0.01	0.94 ± 0.02	0.90 ± 0.01	0.90
4	1.086 ± 0.015	1.085 ± 0.005	1.20 ± 0.05	1.20 ± 0.06	1.13
5	—	1.264 ± 0.002	1.35 ± 0.15	1.56 ± 0.16	1.29
6	—	1.399 ± 0.001	1.40 ± 0.15	1.75 ± 0.25	1.41
7	—	1.5[e]	1.43 ± 0.15	2.10 ± 0.48	1.50

[a] Kurze and Fisher (1979).
[b] Bonfim *et al* (1981) (quoted errors reflect the spread in different extrapolations).
[c] Redner and Yang (1982).
[d] Flory approximation: Redner and Coniglio (1982).
[e] Exact.

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Note added. After this work was completed, I heard that F Family had derived the hyperscaling relation $\theta = (d-1)\nu_\perp$ on general grounds. Also, H E Stanley, S Redner and Z-R Yang have independently proposed the relation $\theta_D = 1 + \sigma(d-1)$ and confirmed this numerically. I thank the above for communicating their results to me.

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